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ABSTRACT

Rate of change has its basis in everyday experience like growth and motion and is a fundamental organizing idea for relationships between varying quantities. In this paper three types of rate of change knowledge for functions are discussed: global, interval, and point-wise. Each of these types of rate of change knowledge can be examined using various function representations including graphs, tables of values, equations, and verbal descriptions. There were two major goals for the larger study from which the smaller study reported here comes. Thirty-seven students at 3 different levels of mathematical experience participated in this study--12 from high school precalculus classes, 15 from second semester college classes, and 10 upper division college mathematics majors. One goal was to describe strategies students use to address global, interval, and point-wise (instantaneous) rates of change. The other goal was to infer how knowledge of one type of rate of change supports students' construction of other types of rate of change knowledge. Findings from this study indicate that high school and college students use changes over intervals to support their thinking in situations represented by graphs and tables of values and use this interval knowledge to address instantaneous rate of change. Contains 17 references. (MKR)

Rate of Change Knowledge in High School and College Students

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Running Head: RATE OF CHANGE

Abstract

Rate of change has its basis in everyday experience like growth and motion and is a fundamental organizing idea for relationships between varying quantities. High school and college students struggle with rate of change tasks. They are often unable to form appropriate algebraic representations of rate of change situations or make correct connections between graphs of amount and rate. But they also often use strategies that point to knowledge that may support their learning rate of change. This knowledge then may serve as a productive cognitive resource in students construction of rate of change knowledge.

Three types of rate of change knowledge for functions are discussed: global, interval, and point-wise. Global rate of change knowledge addresses general properties of a function such as increasing and decreasing and increasing/decreasing at steady, faster, and slower rates. Interval rate of change knowledge concerns the change and average rate of change in the dependent variable over intervals of the independent variable. Point-wise (instantaneous) rate of change knowledge deals with how fast the dependent variable is changing at a value of the independent variable. Each of these three types of rate of change knowledge can be examined using various function representations, including graphs, tables of values, equations, and verbal descriptions.

There were two major goals for the larger study from which the smaller study reported here comes. One goal was to describe strategies students use to address global, interval, and point-wise rate of change. The other goal was to infer how knowledge of one type of rate of change (global, interval, instantaneous) supports students' construction of other types of rate of change knowledge. Of particular interest in the part of the study reported here are the strategies students use to address global, interval, and instantaneous rate of change, and more specifically, students' use of interval rate of change to address instantaneous rate of change. Findings from this study indicate that high school and college students use changes over intervals to support their thinking in situations represented by graphs and tables of values and use this interval knowledge to address instantaneous rate of change.

Introduction to the Problem

Almost every calculus teacher has stories about students' difficulties with calculus. These difficulties are attributed to a number of deficiencies. Students lack knowledge of functions. They have difficulty forming and manipulating algebraic expressions for relationships between quantities, are unfamiliar with the graphs of families of functions and important properties of those graphs, and lack the ability to move comfortably among function representations such as equations, tables of values, and graphs. Two major concepts of calculus are change and accumulation and both deal with functions. It should not then be surprising when students who lack the appropriate function knowledge experience difficulty with calculus.

However, to focus solely on students' difficulties is to miss part of the story of learning. While it is important to know where students struggle with a particular concept, it may be more helpful to know in what ways students are successful in their problem-solving with that concept. Specifically, we would want to know what knowledge students use and how they use it. With regard to rate of change, it would be helpful to know how students use their current knowledge to support their thinking about rate of change situations and to advance their knowledge of rate of change. This information would make teaching and learning calculus in general and rate of change in particular a more fruitful enterprise.

A good place to start this investigation is to see what strategies high school and college students use in rate of change situations. In particular it would be helpful to describe strategies used by three groups of students: ones who have not yet had calculus, ones who are finishing their year of elementary calculus, and ones who have had several mathematics courses beyond calculus. What strategies do these students use? Are there differences in which groups use which strategies and how? Is there evidence for different kinds of rate of change knowledge and, if so, what role do they seem to play in these students' construction of other rate of change knowledge?

Rate of change is an important concept. It has its basis in everyday experiences like motion and growth. Change/rate of change is a fundamental organizing idea for relationships between variable quantities (Confrey and Smith, 1992; Stewart, 1993). It helps us understand and organize our world around different types of change and it helps us predict, and within limits, control our world (Stewart, 1993). The rate of change concept is fundamental to a study of much of theoretical and applied mathematics. It also figures prominently in describing properties of relationships between changing quantities in biology, physics, chemistry, and economics. It seems reasonable then that studies which describe strategies high school and college students use to support their thinking in rate of change situations would yield helpful information about how they build rate of change knowledge, information that would help teachers and curriculum designers facilitate that construction.

Literature on Students' Knowledge of Rate of Change

Students working with graphs of velocity and position functions. Researchers at TERC interviewed high school algebra and physics students using physical models that automatically produce graphs of the position and velocity of cars moving back and forth on a track and graphs of the volume and rate of air being pumped into and out of a bag by a bellows (Rubin and Nemirovsky, 1991; 1992; Monk and Nemirovsky, 1992). In both studies students' predictions about rate or amount graphs given a graph of the other were often incorrect. For example, when given velocity graphs, students gave predicted position graphs which bore incorrect similarities to the velocity graph. Students concluded that if the velocity graph was a line then so was the position graph and if the velocity graph was decreasing then the position graph was also decreasing even when the velocity graph was positive.

McDermott, Rosenquist, and van Zee (1987) reported the trouble college physics and mathematics students experienced in making connections between velocity and position functions represented as graphs. One particular task asked students to use the graph of the position of objects A and B to decide which of the objects was moving faster. Students responded that object B was moving faster than object A since on the graph object B was higher than object A. In actuality object A is moving faster than object B throughout the entire time since the graph for A has the larger slope compared to graph B.

Clement (1989) called this a "height-for-slope" error; students attended to the height of the position graphs when they should have attended to the slopes. In another task Clement presented students with a graph of the velocity of two moving objects. Students used the fact that the cars were at the same point on the velocity graph to conclude that the cars were at the same position. The four studies reviewed above highlight several difficulties high school and college students have when working with rate of change situations represented as graphs. They often assume incorrect resemblances between amount and rate graphs and they use features of one graph to make incorrect conclusions about the other.

Students working rate problems. In a study conducted by Monk (1992a, 1992b) college mathematics majors had difficulty using standard calculus techniques to solve two standard related rates problems. However, several of these students used interval techniques to work these problems. Some noted changes in amount from the beginning of time and changes in amount over smaller time intervals although they often confused the two. Some of the students also had trouble distinguishing changes in one amount with changes in another amount. Despite these difficulties, students' use of changes over intervals to work rate problems is evidence that interval knowledge supports their thinking in rate of change situations and may help them come to distinguish among different quantities in rate of change situations.

Hall (1989) noted the same phenomenon in computer science majors working algebra story problems. While many used standard algebraic techniques, some used alternative methods like generate-and-test and simulation. These alternative methods were equally effective in yielding correct solutions and point to cognitive resources students bring to such problem solving experiences. Monk's students, like Hall's, would not necessarily be considered expert but they would be considered competent in that they had to display a certain amount of mathematical knowledge to get where they are. The strategies that these students bring to rate of change situations may give us information about how those strategies support their problem-solving and advance their knowledge of rate of change.

Thompson (in press) worked with advanced undergraduate and beginning graduate mathematics students on a problem that highlighted their lack of understanding of the intrinsic relationship between the derivative and the integral as expressed in the Fundamental Theorem of Calculus. He concluded that a major source of difficulty for students in learning this important relationship is their "impoverished images of rate" (Thompson, in press).

Students who have a "mature image of rate" see the intimate relationship between accruals (increments, differences) and accumulations. If two quantities accumulate so that their ratio is constant (total distance s/total time t), then their corresponding increments also have the same ratio (delta s/delta t). If two quantities change by increments that are in constant ratio, (delta s/delta t) then their total accumulations have the same constant ratio (total s/total t). "A hallmark of a mature image of rate is that accrual necessarily implies accumulation and accumulation necessarily implies accrual." (Thompson, in press)

The students in Thompson's study struggled with the concept of average rate of change as expressed by (f(x+h)-f(x))/h. When Thompson's students computed it, they did not know how to interpret it. They also did not know what significance there was in computing average rate of change for decreasing values of h. I have observed this in my own students. The definition of instantaneous rate of change of the dependent variable at an individual value of the independent variable is intimately connected to the average rate of change of that quantity over successively smaller intervals of the form (x,x+h) for h positive and (x+h,x) for h negative. If students struggle with the concept of average rate of change then we should not be surprised when they fail to understand the subtlies of meaning in the concept of instantaneous rate of change including an understanding of the Fundamental Theorem of Calculus. Perhaps an important resource in advancing one's knowledge of rate of change is a more complete understanding of average rate of change over intervals and especially over intervals of decreasing width.

Summary of reviewed literature. The studies reviewed here point out two aspects of students' knowledge of rate of change. One aspect deals with the struggles students have in coordinating amount and rate (Rubin, Nemirovsky, Monk, Clement, McDermott, Rosenquist, and van Zee, and Thompson). The other aspect deals with cognitive resources students bring to rate of change situations and how these resources may play an important role in learning rate of change. Students seem to have a propensity for computing changes in the dependent variable from the beginning value of the independent variable and over intervals from one value of the independent variable to another. Perhaps this tendency is a productive resource in learning rate of change. The work of Monk and Thompson point strongly to the possibility that knowing how to deal with changing quantities over intervals holds a key element in our understanding of how students learn rate of change.

Analytical Framework

Function representations. The concept of rate of change deals with changes in one quantity (Y) in relation to changes in another quantity (X). The relationship between two changing quantities can be thought of as a function. The independent variable is quantity X and the dependent variable is quantity Y. So rate of change knowledge is connected to function knowledge and the ways functions are represented. Three function representations figure prominently in thinking about rate of change - graphs, tables, and equations. Each representation highlights different features of rate of change. Appendix E summarizes some of the rate of change features highlighted in each of these three function representations.

What it means to know rate of change. Rate of change is both a deep and wide concept. For example, it has both qualitative and quantitative aspects. Learning qualitative aspects of rate of change involves coming to recognize such features of a function as the dependent variable increasing faster for some values of the independent variable and increasing slower for other values of the independent variable without obtaining, or finding it necessary to obtain, a numerical value that describes those qualities. Learning quantitative aspects of rate of change involves coming to recognize various quantities like total and average rate of change of the dependent variable from the beginning value of the independent variable, change and average rate of change of the dependent variable over an interval of the independent variable, change and average rate of change over intervals of decreasing width around a particular value of the independent variable, and instantaneous rate of change at a particular value of the independent variable.

It is not clear, however, how students learn these qualitative and quantitative aspects of rate of change. As a first step in investigating these matters, I have identified three types of rate of change knowledge that seem to capture qualitative and quantitative aspects of rate of change and take into consideration the kinds of quantities in rate of change situations. I have labeled these global (macro qualitative), interval (macro quantitative), and point-wise (micro quantitative) rate of change knowledge. In the discussion below these names have been shorten to global, interval, and point-wise, respectively.

By global rate of change knowledge I mean the ability to address general properties of a function such as increasing and decreasing and increasing/decreasing at steady, faster, and slower rates. The three function representations display global rate of change of a process in different ways. Continuous well-behaved functions can be represented by graphs which are combinations of seven basic shapes (Nemirovsky, 1991). (See Figures 3 through 9 in Appendix A.) Figure 3 shows a function that is constant. The functions in Figures 4 and 7 increase/decrease at a constant (steady) rate. Figures 5 and 8 show functions that increase/decrease faster and faster. The functions in Figures 6 and 9 increase/decrease slower and slower. Knowing the distinctive properties of each of these seven basic shapes and being able to distinguish among them is part of global rate of change knowledge. Global rate of change features of a function can also be obtained from a table of values and an equation. See Appendix D for a summary table of what it means to know global, interval, and point-wise rate of change and how each type of rate of change knowledge can be obtained from each function representation.

While global rate of change deals with an overall view of the increasing/decreasing behavior of a function, interval rate of change concerns changes and average changes in the dependent variable over intervals of the independent variable. Students can use a graph to address interval rate of change of a quantity by noting the approximate amount of vertical rise or drop of the graph during intervals of the independent variable. In a table of values students can compute the differences between successive values of the dependent variable to address changes over each interval. Students can use an equation to compute the exact change in a dependent variable over an interval by evaluating the equation at the endpoints of the interval and taking the difference. The average rate of change over an interval can also be computed by dividing this difference by the width of the interval.

While interval rate of change addresses change and average rate of change in the dependent variable over intervals of the independent variable, point-wise (instantaneous) rate of change concerns the change in the dependent variable at a particular value of the independent variable. There are several ways to address instantaneous rate of change. If the function is represented by a graph, one can approximate the instantaneous rate of change of the dependent variable for a particular value of the independent variable by estimating the slope of the line tangent to the graph at that particular point. Another perhaps more precise value for the instantaneous rate of change can be obtained from a table of values by computing the average rate of change in the dependent variable over intervals immediately before and after the value of the independent variable. The smaller the width of these intervals, the better the estimate. An even more precise value can be obtained by looking at the limiting pattern formed by computing the average rate of change in the dependent variable over intervals of decreasing width around the value of the independent variable. The most precise value for the instantaneous rate of change can be obtained from an equation by symbolically computing the first derivative and evaluating it for the appropriate value of the independent variable.

Conjectures and goals. Of particular interest in the part of the study reported here are the resources students use to address instantaneous rate of change, specifically features of interval rate of change used to address instantaneous rate of change. Two categories of rate of change quantities will be considered here. One category deals with interval rate of change quantities, notably total change and average rate of change in the dependent variable from the beginning value of the independent variable, and change (increment) and average rate of change in the dependent variable over an interval from one value of the independent variable to another. Knowing interval rate of change knowledge includes the ability to use these different types of interval quantities and to distinguish among them. What interval rate of change quantities do students use in rate of change situations and how do these quantities support their thinking in these situations and advance their knowledge of rate of change?

A second category deals with the point-wise rate of change quantity called the instantaneous rate of change. Knowing point-wise rate of change knowledge includes the recognition of the existence of such a quantity and the ability to find it in several ways. Instantaneous rate of change of a quantity at a particular value of the independent variable is the slope of the line tangent to the graph of the quantity at the point corresponding to that value of the independent variable or the limit of the slopes of secant lines moving in on the point of tangency. This limit can also be thought of as the limit of the average rate of change over intervals of decreasing width around the point. These ways of determining the instantaneous rate of change make use of intervals and will be called interval techniques.

If an equation for the relationship between the dependent variable and the independent variable is available, the instantaneous rate of change can be found by symbolically computing the derivative and evaluating it at the appropriate value of the independent variable. In this study equations are not given and some students have not studied symbolic differentiation. Of interest here are the resources students use to get at instantaneous rate of change when given graphs and tables of values and what this tells us about how students construct knowledge of instantaneous rate of change. It is conjectured that it is knowledge of interval rate of change that serves as a primary resource for that construction.

Methods

Subjects. Thirty-seven students at three different levels of mathematical experience participated in this study - 12 from high school precalculus classes, 15 from second semester college calculus classes, and 10 upper division college mathematics majors. Each group of students was chosen from a larger group of volunteers. All of the high school students came from the same high school and had the same precalculus teacher. Most of the college students came from a small, private liberal arts college; four of the second semester calculus students came from a community college. All three of these schools are in the same county. Males and females were nearly equally represented in each of the three groups. Students in the calculus and post-calculus groups would be expected to know something about instantaneous rate of change. I wanted to know what resources they use to get at instantaneous rate of change and if the post-calculus students use more or different types of techniques than calculus students. I also wanted to see what techniques students who are "ready" to take calculus use to answer questions about instantaneous rate of change.

Data Collection. The students were interviewed individually and were paid for their time. The interviews were audio and video taped. Each interview lasted about an hour and consisted of six tasks. In the first three tasks students were asked to draw a graph and make a table of values showing the distance between two people as they walk toward each other at faster, steady, and slower rates. For the fourth task students were given a graph and a table of values for a ball thrown up into the air and asked to describe the height of the ball over the time, identify intervals where the ball is moving faster and slower, and find the speed of the ball at several points in time. The fifth task is the one discussed in this paper. The sixth task showed a graph and table of values for the relationship between the radius and area of circles formed when a pebble is dropped into a calm pond.

Appendix B contains the fifth task of the interviews. In this task students were presented with a graph and table of values and asked questions designed to investigate their knowledge of global (item a.), interval (item b.), and point-wise (items c., d., e., and f.) rate of change. Of particular interest for items c., d., e., and f. are the resources students use to address instantaneous rate of change.

Data analysis. The video and audio tapes were transcribed and then analyzed in two ways. The first step in analysis involved categorizing the strategies students used to answer each item, deciding types of responses indicating each category, and placing student responses in the categories. The second step in analysis was to look for patterns within students/across items and across students/within items. It was conjectured that students would use interval rate of change knowledge to address instantaneous rate of change at 4am with varying amounts of precision. Of particular interest were the following pieces of interval rate of change knowledge: total change and average rate of change from the beginning of time ($t=0$ (midnight)), change and average rate of change over the interval from 3am to 4am or the interval from 4am to 5am or some combination of these, change and average rate of change over smaller and smaller intervals around 4am or some reference to a limiting process involving these types of intervals. For the most part students gave clear evidence of which of these interval quantities they were using by explicitly noting it or by responding unambiguously to a question about what they were doing. Especially problematic was the distinction between students' use of the interval from $t=0$ to $t=4$ versus their use of the height of the ball at $t=4$ to address the speed of the ball at $t=4$.

Results

Global rate of change knowledge. Students gave one of two responses to the item "Describe the population of yeast cells over this period of time." (item a.). Table 1 in Appendix C shows the responses and the number of students in each group who gave each type of response. The more frequently occurring response was "slower, faster, slower". Twenty-eight of the 37 students gave this type of response. These 28 students were distributed roughly equally across the three groups. Typical student comments were: (Note: The researcher's conversation is in parentheses and the students' gestures are in square brackets.)

> Well, it - first it increases kind of slowly. And then in the middle part of the graph it's increasing faster and then it kind of levels out towards the end. (KG - precalculus)
>
> They increase slowly at the beginning then make huge jumps and then slow down again towards the end. (TW - precalculus)
>
> It takes a little bit for it to start. It increases slow. And then it increases a lot. And then it slows down again. (LC - calculus)
>
> Well, when they start out the first few hours, the yeast are multiplying slowly. And then for some reason they increase rapidly from about 5 to start slowing down up here at about 11 or 12 and then they just level off. (RF - calculus)
>
> It grows a little at first and then it sort of grows a lot more and then it sort of levels off. (DB - postcalculus)
>
> Well, it's going up slow, and then it shoots up real fast, and then it starts going up slow again. (MG - postcalculus)

These students saw the graph as divided into three distinct sections. The first part increases slowly, the second part increases quickly, and the last part increases slowly. In terms of the seven basic shapes for graphs of continuous well-behaved functions, it appears these students saw this graph as a combination of Figures 5, 4, and 6. They gave descriptions that clearly identify three sections of the graph consistent with this combination of figures.

The remaining nine students in the sample saw the graph as divided into two distinct sections. The first half of the graph increased at an increasing rate and the second half increased at a decreasing rate. This group included one precalculus, five calculus, and 3 postcalculus students. Typical comments were:

> My opinion would be they reproduce, I assume that's what's happening , they reproduce and then it's almost like the 2, 4, 8, 16 - like a chain reaction kind of where they're all reproducing until they get to like a certain number, probably when it fills the petri dish and they start slowing down because of absence of space and then they stop producing right there [end of graph]. (JM - precalculus)

> Ok. Well, it's increasing the whole time. And when it first starts out, actually the slope is increasing too. So it's getting bigger faster. Then when it changes here [the point (8,350) on the graph] and starts decreasing, when the slope starts decreasing, then it's getting bigger slower. (BS - calculus)

> Ok. Well from 0 to 8 hours the population is increasing at an increasing rate. And from 8 hours to 18 hours it's increasing but it's increasing at a decreasing rate. (DW - calculus)

> It increases the whole time and it increases more rapidly on this side [left half] than that side [right half]. (Oh, really?) Well, it increasingly increases there so it gets faster for each hour and then here it decreasingly increases. So it gets slower. The population gets bigger at a slower rate for each hour. (JG - postcalculus)

> It starts off very slow. The population is increasing progressively throughout the whole thing. But the rate of change changes throughout the night and the morning and the next day. It starts off slow and as we go during the day it populates faster 'til about 8, after 8 hours. And then it's still growing fast but it decreases slowly and then it really decreases in its populating. (And what is it that's decreasing?) Its rate of how it, of the population. (SE - postcalculus)

For most of these students the halfway point, the point at $t=8$, is the place where the graph changes from increasing at an increasing rate to increasing at a decreasing rate. Several of the calculus students identified this point as the inflection point and referred to the first half of the graph as concave upward and the second half of the graph as concave downward. This was usually in response to a direct question from me about what the name of that point is or what words describe each of those two sections of the graph. In terms of the seven basic shapes for graphs of continuous well-behaved functions, these students appear to have seen this graph as Figure 5 followed by Figure 6 and gave descriptions consistent with that - increasing at an increasing rate followed by increasing at a decreasing rate.

What difference does it make in students' understanding of rate of change in general and global rate of change in particular if they see this graph as increasing slow, fast, then slow or as increasing at an increasing rate then increasing at a decreasing rate? One of the main concepts in differential calculus is concavity and points of inflection. This function is increasing and concave upward in its first half and increasing and concave downward in its second half and has a point of inflection near t=8. This means that the function increases at an increasing rate and then increases at a decreasing rate and changes from one to the other near t=8. Knowing global rate of change means coming to recognize this feature of functions. It should not be surprising then that the students who did were primarily calculus and post-calculus students. An issue yet to be addressed is the role, if any, that seeing this graph as increasing slow, fast, then slow plays in students' coming to know about concavity.

Interval rate of change knowledge. In item b., students were asked to identify intervals in which the function changed the most and the least. All the students successfully named intervals of hours that met these criteria. When asked how they knew, they gave one of two types of responses. Table 2 in Appendix C shows the types of responses and the number of students in each group who gave each type of response. The more frequently occurring response had to do with vertical change in the y-values. Eighteen students gave this type of response - six precalculus, seven calculus, and five postcalculus students. Typical comments include:

> The first four hours, in each of those hours, the number of cells doesn't rise very much. It's still under 100. The number of cells - it's hard to even tell how much it's

going up. But then you get up to about 5 hours until 13 hours and it's going up significantly, between 50 and a 100 per hour and back again to insignificant. (LC - precalculus)

It goes up in a straighter line and goes more distance between boxes [points]. (And by straighter line, you mean?) more vertical. (AJ - precalculus)

Just take the cell population per hour. Here [3 to 4] you have about 25 cells per hour. And then way right here [7 to 8] you have nearly 100 cells per hour. And then, up here [16 to 17 to 18], it's barely changing at all. (RF - calculus)

From 0 to 2, it changes only 48. And from 8 to 10, it changes about - 600 to 800 - 200 more. (RS - calculus)

Here [0 to 1 to 2] the difference would be probably 10 and then as you come up from there [near 3 to 4 to 5] it would be about 40 and from there to there to there [6 to 7 to 8] it's about 100 cells, not quite, and it increases again and here [at the end] it's about 1 cell, 2 cells. (JG - postcalculus)

From zero to 4 it covers say 85; then 4 to 8, it jumps way up to 360. From 85 to 360 is quite a big gap. The population increases much faster in the same amount of time. (RC - postcalculus)

For these students vertical change between consecutive points is the way they are identifying intervals where the change is most and least.

Fourteen students identified intervals where the change is most and least by referring to the slope. These included two precalculus, eight calculus, and four postcalculus students. Typical comments were:

It's more steep [where it's changing most]. It's almost a horizontal line [where it's changing least]. (MH - precalculus)

It's steeper in the middle [where it's changing most]. (JW - precalculus)

The slope is much greater there [5 to 12] than it is here [0 to 4]. (JK - calculus)

Because the slope is greater over here [the middle portion of the graph] then it is here [the beginning and the end]. (PG - calculus)

> The slope of the line is more horizontal here [at the beginning]. It's more vertical here [in the middle]. And then it goes back to horizontal [at the end]. Horizontal slope means that it's not growing very fast and vertical slope means it's growing very fast. (JS - postcalculus)

> Because of the slope. The slope is low here [at the beginning] and the slope is high here [in the middle]. (MG - postcalculus)

Here students identify greater or steeper slope with fastest change and low or horizontal slope with slowest change. I asked some of these students why slope gave information about rate of change. Here are some of their comments:

> Because in this case, over say this hour [8 to 9], there's a greater delta fluctuation [running his pen along the y-axis] so then say in this hour over here [0 to 1] where this distance [y distance] would be much less. (PG - calculus)

> Because over this amount of time [at the beginning] there's very little difference between this point and this point. But then right here [near 6] it starts growing more for a smaller amount of time. (JS - postcalculus)

Both of these students referred to change in number of cells relative to change in time as an explanation for why slope tells how fast the cell population is growing. There is an intimate connection between slope and rate of change. Slope of line and average rate of change over an interval are both defined as change in y over change in x. A typical definition of slope is rise (change in y) over run (change in x). A large majority gave answers that directly referred to slope or the change in y over selected intervals. This represents a powerful way to think about interval rate of change and serves as a clue about how to help students construct knowledge of instantaneous rate of change.

Some may argue that these students are merely showing evidence of school learning in their answers to item b. They probably did learn about slope from school. But the point of this paper is to identify the cognitive resources students bring to rate of change situations and the ways they use those resources to think about and advance their knowledge of rate of change. It will no doubt be the case that students bring many things to such experiences and some of what they bring will be the result of school learning. This is good news. We hope students will use what they have learned in school to think about problems and to advance their knowledge. But we also know that students bring other cognitive resources to these situations and some of these resources represent knowledge they have not learned in school. More pertinent to this paper is how students use the their knowledge, school-acquired or otherwise, to think about and advance their knowledge of rate of change?

Point-wise rate of change knowledge with a graph. For item c., students were to use the graph to answer the question "How fast is the population changing at 4am?". Table 3 in Appendix shows the types of responses and the number of students in each group who gave each kind of response. The most frequent response was the rate of change at $t=4$ is the visible change from 3 to 4. Seven precalculus, three calculus, and two postcalculus students gave this response. Typical comments were:

> About 20 cells an hour because that's how much it changed between 3 and 4. (KH - precalculus)
>
> I would say 25 cells an hour. This looks like 75 [height at 4]. This looks like 50 [height at 3]. So difference is 25 in that duration. (AW - calculus)
>
> It's probably about 75 [at 4] and 50 [at 3] over um 4 minus 3 which is 25 over 1 which is 25 cells added per hour. (LB - postcalculus)

Looking at the change in an interval before 4am is, for these students, a reasonable way of thinking about rate of change at $t=4$. One of the precalculus students made an interesting observation:

> A little faster than 25 yeast cells per hour. At 3, it looks like about 50 and at 4, it looks like about 75. So that's 25, and it's growing faster. So, a little faster. (And you're saying that because it grew 25 cells but you also) It's still going up. (AJ - precalculus)

This student recognized that while the change from 3 to 4 may be a good estimate for how fast the population is changing at 4am, he knows that since it is increasing even more after 4, it has to be growing faster at 4 than it is over the interval from 3 to 4.

For this large group of students, nearly one-third of the entire sample, rate of change over the interval immediately before $t=4$ is used to address instantaneous rate of change at $t=4$. This includes over half (seven) of the precalculus students who would not be expected to use the method most preferred by calculus and postcalculus students, finding the slope of the line tangent to the graph at $t=4$. Eight students (four calculus and four postcalculus) used this technique and half of them noted that the slope of the tangent gives the exact answer to the question but that their computation of the slope of the tangent line was approximate because their tangent line may not have been drawn exactly or their eye-balling of points on their tangent line may have been off a bit.

Five students (two precalculus and three calculus) answered the question about how fast the population is changing at 4am by eye-balling the height at $t=4$ and dividing by four:

> Um - well - there's 80 cells by four hours. So about 20 cells an hour. (JF - precalculus)

> About 17.5 (Why?) 70 divided by 4. (SCL - precalculus)

> Uh, because that's around about 80 cells [at $t=4$] had been developed at that time and it was over a four second [hour] time period. (JP - calculus)

> I would say that they're changing at about 75 [height at $t=4$] feet [cells] per four seconds [hours]. (HL - calculus)

> [She draws a circle around the point at $t=4$ and makes a dashed line left from that point to the y-axis] That's probably about 75 [height at $t=4$; she then writes 75/4 secs on the bottom of her graph]. (SB - calculus)

An alternate interpretation of these students' words may be that they thought the number of cells at $t=0$ is zero and therefore they were actually computing the average rate of change from $t=0$ to $t=4$. But these students' responses here and in item d. show that even when it was pointed out to them that the graph does not start at zero, they still answered the question about how fast the population is changing at $t=4$ by dividing the height at $t=4$, and not the change from 0 to 4, by four. The last student in this group also drew a small line segment tangent to the graph at $t=4$ but then did not do anything with it.

Two postcalculus students computed the average rate of change from $t=0$ to $t=4$ to work item c:

> It went from zero to 75 more cells in four hours. (CH - postcalculus)
>
> It's gone from zero to that point [point on graph at $t=4$]. (RC - postcalculus)

These students believed that the graph shows zero cells at $t=0$ and computed the average rate of change from $t=0$ to $t=4$. They give further evidence of this in their responses to item d.

There are important differences in the techniques of these last two groups of students. The former group of students is displaying what Clement calls a "height-for-slope" error. They are attending to the height at $t=4$ rather than the slope from $t=0$ to $t=4$. The latter group is using the average rate of change over a large interval before 4 and while there may be better estimates for the rate of change at $t=4$, nonetheless it is an interval technique that may be a starting point for students' use of interval rate of change knowledge to construct knowledge of instantaneous rate of change.

Of the 37 students, 28 did not use any of the information about what the graph is doing after t=4 to discuss what is happening at t=4. Nine students (five calculus and four postcalculus) took into consideration what is happening after t=4 as well as what is happening before t=4 to address what is happening at t=4. They computed the average of the changes from 3 to 4 and from 4 to 5 or they divided the change from 3 to 5 by two. Numerically these two methods yield the same answer but may represent different techniques that say something about the state of students' rate of change knowledge. It may be significant to note that four of the five calculus students used the total change from 3 to 5 divided by two while all four of the postcalculus students took the average of the changes from 3 to 4 and 4 to 5. The difference in these two techniques may be inconsequential; both take into consideration what happens immediately before and after 4am and that seems to be the most important knowledge feature of this technique.

The categories discussed above account for all of the postcalculus students' responses to task c. Two calculus students were not included in the categories discussed above. One of these students noted, after computing the changes from 3 to 4 (25) and from 4 to 5 (45), that the rate of change at 4 was "not quite doubling. Just five shy.". This student is not counted in the nine above who used information on both sides of 4 but it seems clear that he believes that what is happening at 4 is somehow affected by what is happening on either side. Another calculus student not counted in any of the categories above drew a very small interval immediately before t=4 and said, "The amount of cell population of this little interval over this little amount of time.". This student is using information about what is happening immediately before t=4 to decide what is happening at t=4.

Three precalculus students gave responses different from those discussed above. One of these students said that she did not know an answer to the question; another answered the question with "It's starting to pick up right there." (BG - precalculus). He noticed that after four it is increasing substantially compared to the increase before 4. A third precalculus student noted that since the number of cells had increased from 75 at 4 to 125 at 5, "It multiplied by two-thirds." This student and the calculus student above who responded that the growth was "not quite doubling" remind me of the work of Confrey and Smith (1992) in noting the power of additive and multiplicative change. Since this function is obviously not additive change, some students expressed their answer in terms of multiplicative change.

In summary, 15 students, half of them precalculus, used what is happening before t=4 to make a judgment about what is happening at t=4; eight, all of them calculus and postcalculus, used the visible slope of the tangent line; 12, most of them calculus and postcalculus, used what is happening both before and after t=4; five, all of them precalculus and calculus), used the height at t=4 divided by four; and one precalculus student did not know how to answer the question.

Point-wise rate of change knowledge with a table of values. For item d., students were given a table of values which they could now use to help them answer the question "How fast is the population changing at 4am?". This is the same question as in as item c. except that students now had both a graph and a table of values for the cell population at each hour. For almost all of the students I made it clear that this table of values represents additional information and tha I was wondering if that helps them think differently about the rate of change at t=4 or confirms their previous answer about the rate of change at t=4. I will examine their responses to item d. as well as compare them with their responses to item c. Table 4 in Appendix C contains the responses to item d. and the numbers of students in each group who gave each response.

Twenty-seven of the 37 students in the sample used the interval immediately before 4 or a combination of the intervals immediately before and after 4. Ten students used the interval immediately before 4. This group contained six precalculus, three calculus, and one postcalculus student. Of all the techniques, this was the technique most preferred by the precalculus students. One of those students noted here as he did in item c. that the answer will be more than the change from 3 to 4 because "after [4] it goes up more". One of the calculus students also indicated that 3 to 4 is better than 4 to 5 "because this [4 to 5] would be an over-estimate because that's [4 to 5] going a lot faster". This is the same student who in item c. answered by referring to a very small interval before t=4. Here she indicated that the change from 3 to 4 was a "major estimate" and that what we really needed was 3.9 to 4.

Eleven students (one precalculus, six calculus, and four postcalculus) took the average of the changes from 3 to 4 and 4 to 5, and six students (no precalculus, three calculus, and three postcalculus) divided the change from 3 to 5 by two. One calculus student noted that the changes from 3 to 4 and 4 to 5 represented the slope of line segments between the corresponding points on the graph and therefore his average of those two slopes represented "a pretty good estimate". One of the postcalculus students remarked that her answer gave the slope of the line through the corresponding points on the graph at 3 and 5 and since this line is pretty much parallel to the line tangent to the graph at t=4, her answer here was better than her answer in c. which she got by eye-balling the slope of the tangent line.

Several of the calculus and postcalculus students offered that their answers here were approximations to the rate of change at t=4 and one postcalculus students noted "for it to be exactly right, your intervals have to be infinitely small. And these are very large, an hour.".

One calculus student said, as he did in item c., that the exact answer is the slope of the tangent line but did not then try to compute it using the two points he had identified on the line. I found out in item e. that he could not remember the formula for the slope of a line. Two postcalculus students computed the average rate of change over the interval from t=0 to t=4. These are the same two students who used this technique in item c.:

> 71.1 cells in a 4 hour period. Oh, but it started at 9.6 [at t=0]. So 71.1 minus 9.6 - that's 62.5. So technically it's 62.5 over four hours. (CH - postcalculus)

> It's gone from 9.6 [at t=0] to 71.1 [at t=4]. So I would say deduct that [9.6] from this [71.1] to give you how fast it changes. (RC - postcalculus)

These two students noticed that the number of cells at t=0 is not zero and used the total change over the four hours rather than simply the height at t=4. Four students (two precalculus and two calculus) divided the height at t=4 by four to answer the question about the rate of change at t=4. Three of these students used the same technique in item c.:

> [I point out to the student that at t=0 the number of cells is 9.6, not zero.] 17.8 [obtained by dividing 71.1, the height at t=4, by four] (JF - precalculus)

> 17.775 [71.1 divided by 4] (SCL - precalculus)

> 71.1 [for four hours] (MY - calculus)

> 71 per at the end of four hours. So per hour, that would be what? [divides 71.1 by 4] 17 cells per hour. (HL - calculus)

Note that I explicitly pointed out to one of these students that the number of cells at t=0 is not zero. She did not seem to take this into account. She proceeded to divide the height at

$t=4$ by the elapsed time. While the data here is sparse and open to multiple interpretations, I believe these students have decided that the speed at $t=4$ is to be found by dividing the height at $t=4$ by the elapsed time.

What accounts for students' use of this technique? Average rate of change in amount Y over an interval of amount X is the change in amount Y divided by the corresponding change in amount X. The average rate of change in Y from $X = x_i$ to $X = x_j$ is given by the ratio $(y_j-y_i)/(x_j-x_i)$. So in the task the students were working on, the average rate of change from $t=0$ to $t=4$ is $(71.1-9.6)/(4-0) = 61.5/4 = 15.375$ cells per hour. Students who divide the height at $t=4$ by 4 are using the ratio $y_j/x_j = 71.1/4 = 17.775$ to address rate of change at $t=4$. The two ratios (delta y/delta x and y/x) are sufficiently similar to almost invite confusion and especially so if they have not been explicitly distinguished in the minds of the students. In addition, in many rate of change situations, the beginning values of X and Y (x_i and y_i) are zero so that the average rate of change of Y from $X = x_i$ to $X = x_j$ turns out to be the same as y_j/x_j. Unless these subtle distinctions are made in the minds of students, it is small wonder that they use y/x when they should be using delta y/delta x.

Three students have not been accounted for in the above discussion. They are the same students who in item c. either did not know, thought it was "starting to pick up right there", or "multiplied by two-thirds". They gave the same responses to item d.

In summary, 12 students, half of them precalculus, used intervals before $t=4$; one calculus student used slope of tangent line; 19 students, most of them calculus and postcalculus, used intervals before and after $t=4$; four students, precalculus and calculus, divided the height at t=4 by four; and one precalculus student did not know. Table 5 in Appendix C shows how these numbers compare to those in item c.

Comparison of responses about point-wise rate of change at t=4 when using the graph and the table of values. To answer the question "How fast is the population changing at 4am?" twenty-two of the 37 students used the same technique for both the graph (item c.) and the table of values (item d.). Table 6 in Appendix C compares responses to items c. and d. Fifteen of the remaining 17 students worked item d. using a technique that should yield an equal or better approximation. For example, three students who used the interval 3 to 4 to work item c. used both of the intervals from 3 to 4 and 4 to 5 (a better approximation) for item d. All of the students who used the slope of the tangent line in item c. worked item d. using the same technique or changes over the interval from 3 to 4 and 4 to 5 (an equally good approximation). The remaining two students included the calculus student who in item c. drew a very small interval immediately before t=4 and wanted in item d. to use the interval from 3.9 to 4. She ended up using the interval from 3 to 4 to get an answer in item d. The other student was the calculus student who for item c. compared the change from 3 to 4 and 4 to 5 and concluded that it was "not quite doubling". He obtained his answer for item d. by dividing the height at t=4 by four.

Point-wise Rate of change at t=4 using an enlarged graph and a more detailed table of values. In item e. students were given an enlargement of the first half of the graph and a table of values from t=3 to t=5 in 15 minute intervals. They were told that this additional information may now be used to give a different answer or confirm a previous answer to the question "How fast is the population changing at 4am?". Table 7 in Appendix C shows the students' responses to item e. Twenty-one students used the intervals immediately before and after t=4 to answer this question. This included 11 of the 15 calculus students and nine of the 10 postcalculus students but only one of the 12 precalculus students. Thirteen of these 21 students averaged the changes from 3.75 to 4 and 4 to 4.25 and then multiplied that average by 4 to get cells per hour. Eight of these 21 students computed the change from 3.75 to 4.25 and then multiplied by 2 to get cells per hour. These students considered what happens on both sides of t=4 in deciding what is happening <u>at</u> t=4. I did not specifically ask students why they believed that was necessary. But it seems clear from their comments that they felt that it was:

> I think I'm going to look at the difference between 3.75 and 4 and 4.25 and 4. And that - and all that together, see, is gonna give me what exactly is going on right at 4. (RC - postcalculus)

In contrast, 10 students (six precalculus, three calculus and one postcalculus) computed the change from t=3.75 to t=4 and multiplied that by four. These students have considered only what is happening before t=4 as relevant to what is happening at t=4. When I asked one of the precalculus students if what is happening immediately after t=4 had any bearing on what is happening at t=4, he gave these comments:

> No, because that's the fifth second [hour] or that's the way I see it. See, that's the fifth second [hour]. So if when you tell me what's happening at 4, I would think all in the fourth second [hour] would be the ones that are included in 4 and then after that would be included in 5. (JM - precalculus)

Of the remaining six students, one calculus student computed the changes from 3.75 to 4 and 4 to 4.25 and concluded "It's on an upward growth." The other five students were precalculus students. One student said she did not know; one said "It's starting to pick up at that point [t=4]." (BG - precalculus); and one computed the change from 4 to 4.25, divided that by 71.1 (the height at t=4), and concluded that there was a 12 percent increase. These three students used the same techniques in items c. and d. The other two precalculus students included one student who said that this new graph and table of values did not change his answer to item d. where he computed the change from 3 to 4. And it included the only student who answered item e. with the height at t=4 divided by four. This same student answered items c. and d. the same way. The other students who worked items c. or d. using the height at t=4 divided by four moved in items d. and e. to using the interval immediately before t=4 or a combination of the intervals immediately before and after t=4.

When students were asked which of their answers they had the most confidence in, almost without exception they said they had more confidence in their answer in e. because the intervals were smaller and what is happening closer to 4 is more relevant to what is happening at t=4 than what is happening farther away from 4:

> (Which of these three answers do you have the most faith in? 20, 35.95, or 33 cells per hour?) 33 (Why?) Because it's going to be more accurate because when you're doing it between 3 and 4, the change in rate over that time is greater than the change in rate just in a quarter of an hour and so there's a bigger margin for error if you're doing 3 and 4 and 4 and 5. (KH - precalculus)

> (So now you've got 31.6 cells per hour and 23.9 cells per hour. Which of these answers do you have more confidence in?) The second one [pointing to 31.6]. (Why is that?) Uh. Well, it has - well, the interval is - it's smaller interval, so it'd be - um - be a more accurate number. (JW - precalculus)

> (Ok. So here we got 33 cells per hour and the other one was 36 cells per hour. So which one of those is better?) I think this [33] is a more precise way of doing it because you're getting closer to an interval of 4, at 4 hours. Before I was just taking the difference here, between 3 and 4, and figuring out how many cells per hour in between here and here [4 to 5]. And I'm kinda zooming in on it [circles entries in table for 3.75, 4, and 4.25] like we're doing here [pointing to the more detailed graph] trying to get more precise. (AW - calculus)

I would think this would be more accurate because we've got smaller intervals of time. (JK - calculus)

(You have more faith in this?) Yes, because this is right next, right around my time, 4. This is much closer than an hour on each side. This is only 15 minutes. (RF - calculus)

(Which one do you think is the better answer and why?) I think 33 cells per hour is probably a better answer because as the time interval gets smaller, the rate of change becomes more accurate. (TS - postcalculus)

(Which answer do you have more faith in?) 33 because it's closer. (CH - postcalculus)

(Which one of these three answers to you have more confidence in?) I have more faith in this one [31.6] because it's closer to t=4. (LB - postcalculus)

Most students declared their answers in c., d., and e. to be approximations to the exact answer.

Getting a better approximation and getting an exact answer. In item f. I asked students how they would get a better approximation and how they would get the exact answer. (See Table 8 in Appendix C.) All of the postcalculus, two-thirds of the calculus, and half of the precalculus students answered that smaller intervals or intervals closer to t=4 would yield a closer approximation. Here we see evidence that students are thinking about the need to get as close to t=4 as possible to get better approximations. Since the definition of instantaneous rate of change of this function at t=4 is based on the average rate of change over intervals of decreasing width around t=4, knowing that one can get a better approximation by computing the average rate of change over intervals of smaller width around t=4 is an important step in coming to know instantaneous rate of change at a point. Clearly all of the postcalculus and most of the calculus students recognize this. It is interesting to know that half of the precalculus students see this. The direction of the percentages (50% for precalculus, 67% for calculus, and 100% for postcalculus) indicates a possible developmental stream for coming to know instantaneous rate of change.

When I asked students how they would get an exact answer, there were three main types of answers. Seventy percent of the postcalculus students and 20 percent of the calculus students said that if they had an equation for this function, they would find the derivative symbolically and evaluate it at $t=4$. Seventy percent of the postcalculus and 27 percent of the calculus students indicated they could get an exact answer by finding the limit of the average rate of change over intervals of decreasing width around $t=4$. Finally about half of the precalculus and calculus group said if they had the exact value of the slope of the tangent line, then that would be the exact value of the instantaneous rate of change at $t=4$.

For precalculus students the question about an exact value for the rate of change at $t=4$ presented special problems. Five (42 percent) of these students gave general answers:

> It changes. (KG - precalculus)

> Before 4 it start increasing and it keep increasing. (SCL - precalculus)

> It's still going about the same. [He estimated 32 feel per second over the interval 3.75 to 4 and 32 feet per second over the interval 4 to 4.25. So he may believe that since these values are equal, then he has found an exact value - 32 feel per second]. (AJ - precalculus)

> [He obtains 32 feet per second for the interval from 3.75 to 4 and 32 feel per second for the interval from 4 to 4.25] It wouldn't change it. (TW - precalculus)

> No - I just - you know - I say - it's starting to pick up at that point [t=4]. (BG - precalculus)

The other seven students indicated they would just take smaller intervals. The tone of their words does not seem to me to imply that they are thinking of a limiting process so much as they are thinking about getting approximations that are simply better.

One question of interest to some is whether or not students believe in instantaneous rate of change. Thinking about average rate of change (change in distance divided by change in time, for example) is a fairly common concept in everyday life. But rate of change at a point is anomalous since rate of change implies change in distance over change in time but rate of change at a point implies no change in distance and no change in time. In these interviews students were presented an earlier problem in which this issue came up. But interestingly it was calculus and postcalculus students who raised it. The precalculus students did not explicitly raise it. They seemed to respond to my requests to find an exact value with attempts to do so without questioning whether or not such a thing exists. I attribute this behavior to the "teacher request" syndrome - "If a teacher asks me to find something then obviously it exists and therefore I will simply try to find it to the best of my ability even if it doesn't make sense to me."

Another interesting feature of students' responses to instantaneous rate of change is that a good many calculus students were quite tentative about how to find it. Note that only 20 percent (3 out of 15) of these students identified instantaneous rate of change at $t=4$ with evaluating the derivative at $t=4$. Only 27 percent (4 out of 15) articulated some kind of limiting process. Only half mentioned the slope of the tangent line. This is in contrast to the postcalculus students 70 percent of whom discussed evaluating the derivative at $t=4$ and explicitly outlined a limiting process for finding the instantaneous rate of change at $t=4$. Again about half mentioned the slope of the tangent line.

Although postcalculus students would be expected to have more experience with the derivative, there are not very many places in the upper division mathematics curriculum where the conceptual underpinnings of the derivative are explicitly part of the instruction. After all, that is the purpose of the elementary calculus course. So why do more postcalculus students show knowledge of instantaneous rate of change than the calculus students. There are several possible reasons. One is that experience with the derivative without explicit consideration of the conceptual underpinnings somehow helps students understand instantaneous rate of change better. Another is that learning follows instruction at a distance longer than unit and final exams. Students may very well have their first encounter with instantaneous rate of change in calculus, but something happens in the months after calculus instruction ends that helps students construct further knowledge of this concept.

Conclusion

The study reported here focusses on the resources students use to get at instantaneous rate of change. A major finding is that for these students interval rate of change was a powerful way to think about instantaneous rate of change. Students used intervals around $t=4$ to address rate of change at $t=4$. Calculus and postcalculus students most often took into account what is happening both immediately before and after $t=4$ to address what is happening at $t=4$ while precalculus students were more inclined to use only intervals before $t=4$. This was most easily done with a table of values. But even with a graph students used intervals by simply estimating y values for certain x values. This is a strong finding.

Students have a tendency to use interval rate of change knowledge to get at rate of change in general and instantaneous rate of change in particular. This ought to be taken into consideration in teaching and curriculum design. The concept of instantaneous rate of change should be approached through interval rate of change. This means that instruction in average rate of change should take place much earlier than elementary calculus. The concept of average rate of change is important in and of itself. That is reason enough for its inclusion in all levels of high school mathematics instruction and curriculum. But an additional reason is that average rate of change is the window through which students initially view instantaneous rate of change and hence instruction on average rate of change will more adequately prepare students for learning instantaneous rate of change.

A second major finding of this study is that students continue to construct knowledge of rate of change after their instruction in elementary calculus is ended. We do not have direct knowledge in this study about further construction of rate of change knowledge in students who do not continue to study mathematics. We do have a fairly strong indication that in students who continue their study of mathematics beyond calculus the construction of knowledge of instantaneous rate of change advances. How this happens or why are questions remaining to be answered. But it seems clear that it does happen. Further research is needed to describe mechanisms for this learning.

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Appendix A

Figures 3 - 9

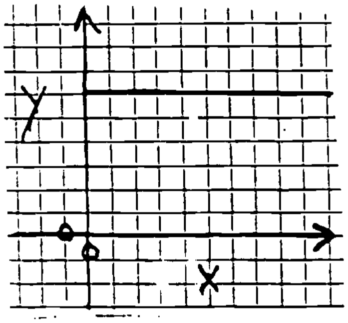

Figure 3

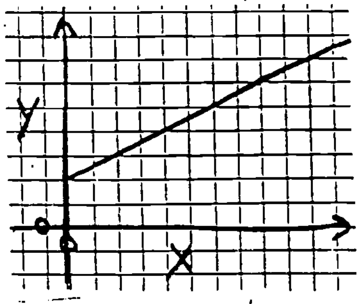

Figure 4

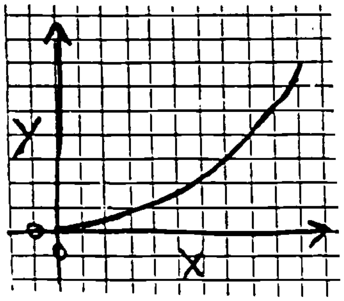

Figure 5

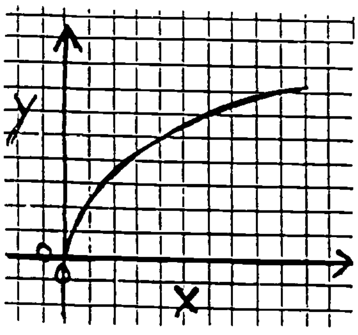

Figure 6

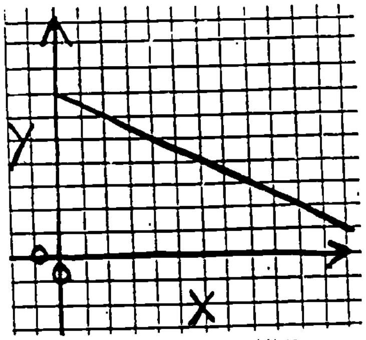

Figure 7

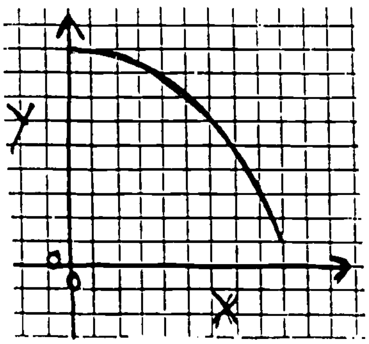

Figure 8

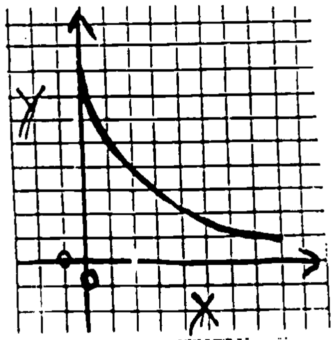

Figure 9

Appendix B

The Task for this study

was constructed using data found in

Giordano and Weir (1985).

They attribute the data to Pearl (1927).

Graph 2 shows the population of yeast cells in a culture over a period of 18 hours (from midnight to 6pm).

a. Describe the population of yeast cells over this period of time.

b. Does the population of yeast cells change about the same each hour or does it change more some hours and less other hours? How do you know?

During which hour(s) does the population of yeast cells change the most?

How do you know? How fast does it change over those hour(s)?

During which hour(s) does the population of yeast cells change the least?

How do you know? How fast does it change over those hour(s)?

During which hours does the population of yeast cells change about the same?

How do you know? How fast does it change over each of these hours?

c. How fast is the population changing at 4am? How do you know?

d. Table 2 shows the values for the data points in Graph 2. Please use it and the graph to answer the same two questions in c.: How fast is the population changing at 4am? How do you know?

e. Graph 3 is a blownup section of graph 2 and table 3 gives values of the yeast cell population for time value in this section. Use these to find out how fast the population is changing at 4am.

f. How much does the population change over the interval from 3am to 4am?

How fast does the population change over the interval from 3am to 4am?

How much does the population change over the interval from 4am to 5am?

How fast does the population change over the interval from 4am to 5am?

How fast does the population change at 4am?

How much does the population change over the interval from 3:30am to 4am?

How fast does the population change over the interval from 3:30am to 4am?

How much does the population change over the interval from 4am to 4:30am?

How fast does the population change over the interval from 4am to 4:30am?

How fast does the population change at 4am?

How much does the population change over the interval from 3:45am to 4am?

How fast does the population change over the interval from 3:45am to 4am?

How much does the population change over the interval from 4am to 4:15am?

How fast does the population change over the interval from 4am to 4:15am?

How fast does the population change at 4am?

Describe a procedure for finding exactly how fast the population changes at 4am.

Graph 2

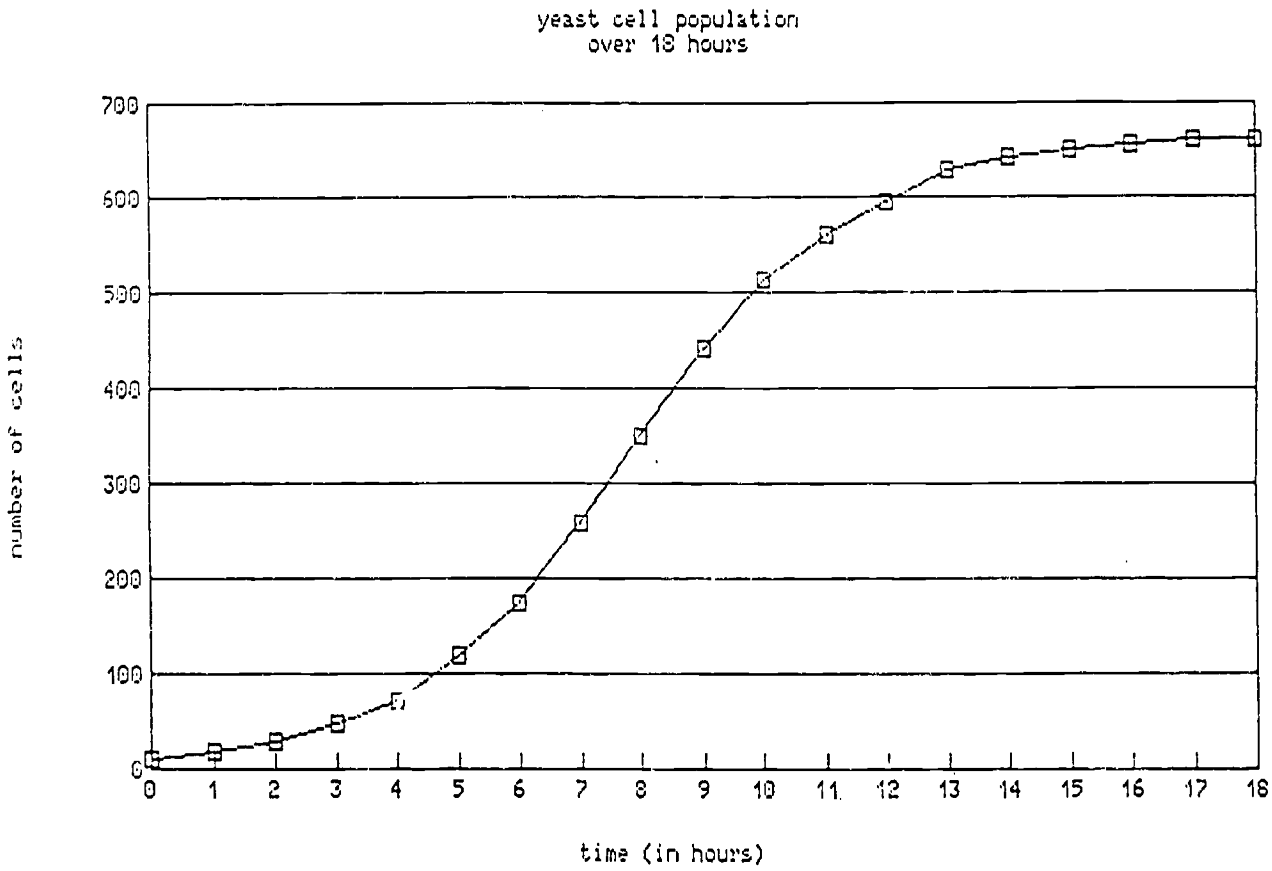

Table 2

Time	Yeast Cell Population
0	9.6
1	18.3
2	29
3	47.2
4	71.1
5	119.1
6	174.6
7	257.3
8	350.7
9	441
10	513.3
11	559.7
12	594.8
13	629.4
14	640.8
15	651.1
16	655.9
17	659.6
18	661.8

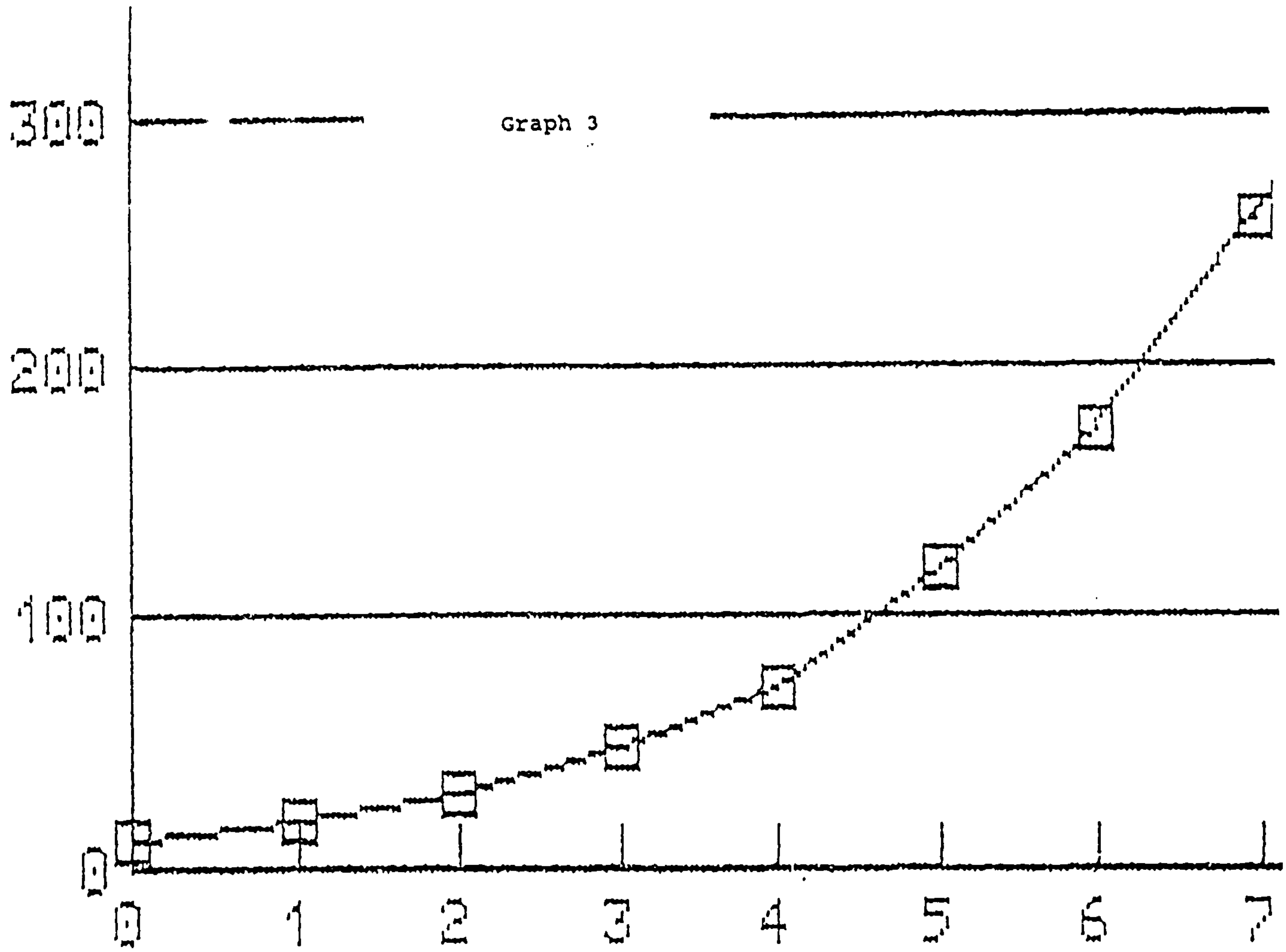

Table 3

Time	Yeast Cell Population
3	47.2
3.25	51.4
3.5	56.7
3.75	63.2
4	71.1
4.25	79.7
4.5	90.3
4.75	103.2
5	119.1

Appendix C

Table 1 - Responses to item a.

Number of students in each response category

Type of response	Precalculus	Calculus	Postcalculus
slower, faster, slower	11	10	7
increasingly increasing, decreasingly increasing	1	5	3
Total:	12	15	10

Table 2 - Responses to item b.

Number of students in each response category

Type of responses	Precalculus	Calculus	Postcalculus
vertical change, diagonal distance	6	7	5
slope/steepness	2	8	4
not asked	4	0	1
Total:	12	15	10

Table 3 - Responses to item c.
Number of students in each response category

Type of response	Precalculus	Calculus	Postcalculus
change from 3 to 4	7	3	2
slope of tangent line	0	4	4
height at 4, divided by 4	2	3	0
3 to 4, 4 to 5, average	0	1	4
3 to 5, divided by 2	0	4	0
0 to 4, divided by 4	0	0	2
other	3	2	0
Total:	12	17*	12*

* Totals for calculus and postcalculus students do not add to 15 and 10, respectively because some students gave more than one response.

Table 4 - Responses to item d.
Number of students in each response category

Type of response	Precalculus	Calculus	Postcalculus
change from 3 to 4	6	3	1
slope of tangent line	0	1	0
height at 4, divided by 4	2	2	0
3 to 4, 4 to 5, average	1	6	4
3 to 5, divided by 2	0	3	3
0 to 4, divided by 4	0	0	2
other	3	0	0
Total:	12	15	10

Table 5 - Comparison of responses to items c. and d.
on the basis of numbers of students in each category

Type of response	Item c. (graph)	Item d. (table of values)
Intervals before t=4	15 (half precalc)	12 (half precalc)
Intervals before and after	12 (most calc and post)	19 (most calc and post)
Slope of tangent line	8 (calc and post)	1 (calc)
Height at t=4 divided by four	5 (precalc and calc)	4 (precalc and calc)
Did not know	1 (precalc)	1 (precalc)
Total:	41*	37

* Some students gave more than one response to item c.

Table 6 - Comparison of Responses to items c. and d.
on the basis of responses of individual students

Answer to task c.	Same response to task d.			Different response		
	precalculus	calc	post	precalculus	calc	post
change from 3 to 4	6	1	1	1	2	1
slope of tangent line	0	1	0	0	3	4
height at 4, divided by 4	2	1	0	0	2	0
3 to 4, 4 to 5, average	0	1	3	0	0	1
3 to 5, divided by 2	0	1	0	0	3	0
0 to 4, divided by 4	0	0	2	0	0	0
other	3	0	0	0	2	0
Total:	11	5	6	1	12	6*

*This includes the postcalculus student who gave three different answers to item c. but gave only one of those answers in item d.

Table 7 - Responses to item e.
Number of students in each response category

Type of response	Precalculus	Calculus	Postcalculus
3.75 to 4, 4 to 4.25	1	7	5
3.75 to 4.25	0	4	4
3.75 to 4	6	3	1
height at 4	1	0	0
other	4	1	0
Total:	12	15	10

Table 8 - Responses to item f.
Number of students in each response category

Type of Response	Precalculus	Calculus	Postcalculus
Use intervals closer to t=4 to getter a better approximation	6 (50%)	10 (67%)	10 (100%)
To get an exact answer:			
Take derivative of equation and evaluate at t=4	0 (0%)	3 (20%)	7 (70%)
Limit of average rate of change over intervals of decreasing width around t=4	0 (0%)	4 (27%)	7 (70%)
Exact value of the slope of the tangent line at t=4	0 (0%)	7 (47%)	6 (60%)

Appendix D

What it means to know rate of change

	Global	Interval	Point-wise
Using graphs	Use the shape and trend of a graph to specify where a function is increasing/decreasing and where it is increasing/decreasing at faster, slower, and steady pace.	Use vertical rises and falls or visible slopes of secant lines to specify intervals where a function is inc/dec and where it is inc/dec at faster, slower, and steady pace.	Eye-ball slope of line tangent to a graph at a particular point to estimate the instantaneous rate of change at that point.
Using tables	Use the values in a table to specify where a function is inc/dec and use differences to specify where a function is inc/dec at faster, slower, and steady pace.	Use table of values and size and direction of differences to specify the amount and type of change in a function over an interval.	Use average rate of change over intervals before and after a particular point to estimate the instantaneous rate of change at that point.
Using equations	Use algebraic/qual. properties of equations to specify where a function is inc/dec and where it is inc/dec at faster, slower, and steady pace. Use f' and f'' to specify where a function is inc/dec and where it is inc/dec at faster, slower, and steady pace.	Compute $f(x_i)$ and $f(x_{i-1})$ to compute changes over an interval and to compute the average rate of change over an interval.	Use an equation to obtain more accurate estimates of inst. rate of change by computing average rate of change over intervals of decreasing width; Use f' to exactly compute the inst. rate of change at a particular point.

Appendix E

Features of relationships between varying quantities displayed by graphs, tables of values, and equations

	Graphs	Tables of values	equations
can be used to decide when a quantity is increasing, decreasing, or staying the same	X	X	X
can be used to specify time intervals when a quantity is increasing or decreasing faster, slower, or at a steady rate	X	X	X
can be used to specify approximately how much quantity changes over a time interval	X	X	X
can be used to specify exactly how much the quantity changes over a time interval		X	X
can be used to find an estimate of the instantaneous rate of change of quantity at a particular point	X	X	X
can be used to find the exact value of the instantaneous rate of change of quantity at a particular point			X